13 Detector and Channel Coding

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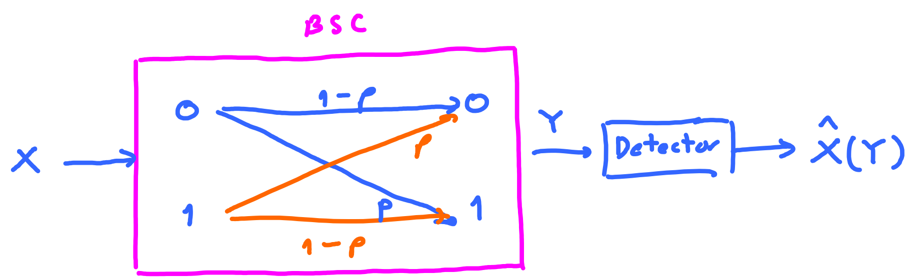
13.1 Detector for BSC



Observe that we can skip the detector and simply say $\hat{X}(Y) = Y$.
This may happen if the p from the BSC is small enough and you don't want to add more complexity to your system.
In which case, your BER would be p.

Observe that if $p = 1$, then you can improve the BER easily by using a detector that performs the flip of bits : $\hat{X}(Y) = \bar{Y}$.
In which case, your BER would be $1-p = 0$.

The detector which we add in this section will try to improve the BER by taking into account both crossover probability and the prior probability of X.

Observe that if you always transmit $X=0$ (and never transmit $X=1$), then it would not be benifical to answer $\hat{X} = 1$ regardless of the observed value of Y. In which case, you should use $\hat{X}(Y) \equiv 0$.

The cases considered above are extreme and quite easy to see which decoder would perform well. In general, there are many detectors that you can use. The best one is called

the MAP detector

which will be discussed below.

Assume $p = 0.1$
crossover probability for the BSC

This gives you four conditional probabilities:

$P[Y=0 \mid X=0] = 1-p = 0.9$

$P[Y=1 \mid X=0] = p = 0.1$

$P[Y=0 \mid X=1] = p = 0.1$

$P[Y=1 \mid X=1] = 1-p = 0.9$

Objective: Want to guess the value of X from Y.

Of course, you want to guess correctly.

In other words, you want to maximize the probability that your guess would be correct.

The optimal guessing is as followed:

If you have not observed Y, then

guess x that maximizes $p_X(x)$.

prior probability

a priori probability

When Y is available, guess x that maximize

$p_{X|Y}(x|y) = P[X=x \mid Y=y]$

posterior probability

a posteriori probability

This is called MAP detector.

maximum a posteriori

Suppose $p_X(x) = \begin{cases} 0.95, & x=1 \\ 0.05, & x=0 \\ 0, & \text{otherwise} \end{cases}$

p_1, p_0

(Alternatively, we can simply write $X \sim \text{bernoulli}(0.95)$.)

Note that the input of the detector is Y which can be 0 or 1.

So, we need to do the analysis two times:

what is the best guess of X value when Y = 1 is observed

and " " " " Y = 0 " .

Suppose Y = 1 is observed, which event is more likely

X = 0 was transmitted or

X = 1 was transmitted ?

$P[X=0 \mid Y=1]$ $\qquad$ $P[X=1 \mid Y=1]$

Bayes' rule/theorem

$= \dfrac{P[Y=1 \mid X=0]\,P[X=0]}{P[Y=1]}$ $\qquad$ $= \dfrac{P[Y=1 \mid X=1]\,P[X=1]}{P[Y=1]}$

$= \dfrac{p \times p_0}{P[Y=1]} = \dfrac{0.1 \times 0.05}{P[Y=1]} < = \dfrac{(1-p)\,p_1}{P[Y=1]} = \dfrac{0.9 \times 0.95}{P[Y=1]}$

$\Downarrow$

$\hat{X}_{MAP}(1) = 1$

Because $P[X=1 \mid Y=1] > P[X=0 \mid Y=1]$, when Y = 1 is observed, it is more likely that X = 1 was transmitted.

So, the MAP detector would guess $\hat{X}_{MAP}(1) = 1$.

Suppose Y = 0 is observed, which event is more likely

X = 0 was transmitted or

X = 1 was transmitted ?

(Equivalently, I may ask $\hat{X}_{MAP}(0) = ?$)

Again, we compare the two a posteriori probability

$P[X=0 \mid Y=0]$ $\qquad$ $P[X=1 \mid Y=0]$

$= \dfrac{P[Y=0 \mid X=0]\,P[X=0]}{P[Y=0]}$ $\qquad$ $= \dfrac{P[Y=0 \mid X=1]\,P[X=1]}{P[Y=0]}$

$= \dfrac{0.9 \times 0.05}{P[Y=0]} \underset{=}{<} = \dfrac{0.1 \times 0.95}{P[Y=0]}$

$$= \frac{0.9 \times 0.05}{P[Y=0]} \quad < \quad = \frac{0.1 \times 0.95}{P[Y=0]}$$

$$\Downarrow$$

$$\hat{X}_{MAP}(0) = 1$$

Conclusion, when $p = 0.1$ and $p_1 = 0.95$,

$$\hat{x}_{MAP}(y) \equiv 1$$

Using this MAP detector, the BER or the error probability is

$$P(\mathcal{E}) = P[\hat{X} \neq X] = p_0 = 0.05$$

For comparison, suppose we use a different detector, say $\hat{x}(y) = y$.

Then, $P(\mathcal{E}) = P[\hat{X} \neq X] = p = 0.1 > 0.05$

$\Downarrow$

So, MAP detector is better.

This means we haven't applied any coding discussed in 13.2.

Def. A MAP detector for uncoded BSC is given by

$$\hat{X}_{MAP}(y) = \arg\max_{x} P[X = x \mid Y = y]$$

13.2 Block Coding